The Complexity of Vector Spin Glasses

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We study the annealed complexity of the m-vector spin glasses in the Sherrington-Kirkpatrick limit. The eigenvalue spectrum of the Hessian matrix of the Thouless-Anderson-Palmer (TAP) free energy is found to consist of a continuous band of positive eigenvalues in addition to an isolated eigenvalue and $(m-1)$ null eigenvalues due to rotational invariance. Rather surprisingly, the band does not extend to zero at any finite temperature. The isolated eigenvalue becomes zero in the thermodynamic limit, as in the Ising case ($m=1$), indicating that the same supersymmetry breaking recently found in Ising spin glasses occurs in vector spin glasses.

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The recent renewed interest in the complexity of spin glasses has produced new insights into their properties [1, 2, 3]. The 1980 calculation of the complexity of the Ising Sherrington-Kirkpatrick (SK) model by Bray and Moore (BM) [4] has been confirmed but new features of it have now emerged. For that model the complexity is the average number of solutions, $\langle N_S\rangle_J$, of the Thouless-Anderson-Palmer (TAP) equations [5]. The Hessian matrix associated with these solutions (the matrix $\partial^2 F/\partial m_i \partial m_j$), where F is their free energy and $m_i = \langle S_i\rangle$ is the local magnetization has to have no negative eigenvalues for a solution of the TAP equations to be a minimum of the TAP free energy. The eigenvalues of this matrix have been found to form a band such that the smallest eigenvalue in the band is positive, together with an isolated eigenvalue which is zero in the large N limit, where N is the number of sites. For N large, the solutions of the TAP equations occur in pairs, with one solution corresponding to a minimum of the free energy and a small positive value for the isolated eigenvalue and the other solution corresponding to an (index-one) saddle point of the free energy F, with the isolated eigenvalue being small but negative. The appearance of a null eigenvalue in the large N limit can be attributed to the breaking of a supersymmetry [3, 6, 7, 8, 9] – the BRST supersymmetry [10] – possessed by the action that appears in the evaluation of $\langle N_S\rangle_J$ [11].

To date only the complexity of the spherical p-spin model has been studied in as much detail as that of the Ising SK model. For this model the supersymmetry is unbroken and so the isolated eigenvalue is no longer a null eigenvalue [12]. The minima are genuine mimima of the free energy and not just turning points as in the Ising case. The dynamical consequences of this difference in the value of the isolated eigenvalue are striking. For the Ising model, the TAP states, being turning points only of the free energy, do not trap the system near them for any substantial period of time, whereas in the spherical p-spin model the system evolves from its initial state and becomes trapped for ever near the TAP solution accessible from the initial state [13].

In this paper, we examine the complexity of models which are often closer to experimental spin glasses than the Ising spin glass, namely, the m-vector spin glass models. They include the XY ($m=2$) and Heisenberg ($m=3$) spin glasses. We studied also the eigenvalue spectrum of the Hessian matrix of the TAP free energy for this model. This was previously obtained at zero temperature in Refs. [14, 15]. We show below that the eigenvalue spectrum at *finite* temperature consists of a continuous band of positive eigenvalues, together with $(m-1)$ null eigenvalues due to rotational invariance and a further null eigenvalue (in the thermodynamic limit) whose origin is as for the Ising spin glass [1]. We had expected that the band edge would be at zero due to the rotational invariance in these models, as found at zero temperature, but in fact there is a small but finite gap when $0 < T < T_c$.

The m-vector SK spin glass model has Hamiltonian

$$\mathcal{H} = -\frac{m}{2}\sum_{ij} J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j, \tag{1}$$

where the m-component spins $\mathbf{S}_i = \{S_i^\alpha\}$, ($\alpha = 1,\ldots,m,\ i=1,\ldots,N$) of unit length $S_i = 1$, and the interactions J_{ij} have a Gaussian distribution with zero mean and variance $1/N$ so that $T_c = 1$ for all m. The TAP equations for the local fields $\mathbf{h}_i$ at temperature $T=\beta^{-1}$ are given in terms of the local magnetization variables $\mathbf{m}_i = \langle\mathbf{S}_i\rangle$ by [14]

$$h_i^\alpha = \sum_j J_{ij} m_j^\alpha - \beta(1-q)m_i^\alpha, \tag{2}$$

where $q = N^{-1}\sum_i \mathbf{m}_i^2$, and $\mathbf{m}_i$ is related to $\mathbf{h}_i$ by $m_i^\alpha = \widehat{h}_i^\alpha L(m\beta h_i)$. The function $L(x) = I_{\frac{m}{2}}(x)/I_{\frac{m-2}{2}}(x)$ and $I_\nu(x)$ is a modified Bessel function. Here $\widehat{\mathbf{h}}_i = \mathbf{h}_i/h_i$. The TAP equations are obtained by extremizing the free

energy

$$F = -\frac{m}{2}\sum_{i,j} J_{ij}\mathbf{m}_i\cdot\mathbf{m}_j - \frac{m}{4}N\beta(1-q)^2 - T\frac{\partial}{\partial T}\left[T\sum_i \ln z(m\beta h_i)\right] \tag{3}$$

with respect to m_i^α, where $z(x) = 2\pi^{\frac{m}{2}} I_{\frac{m-2}{2}}(x)/(\frac{x}{2})^{\frac{m-2}{2}}$ [15]. Using the TAP equations, (2), the free energy can be written as $\beta F/N = N^{-1}\sum_i f(m\beta h_i)$ where

$$f(x) = -\frac{1}{4}m\beta^2(1-q^2) + \frac{x}{2}L(x) - \ln z(x). \tag{4}$$

Following [4, 14], the number $N_S(f)$ of solutions to the TAP equations with free energy f per spin is given by

$$N_S(f) = N^2\int dq \int \prod_{i,\alpha} dm_i^\alpha \int \prod_{i,\alpha} dh_i^\alpha |\det \mathbf{K}| \tag{5}$$
$$\times \prod_{i,\alpha}\delta\Big(h_i^\alpha - \sum_j J_{ij}m_j^\alpha + \beta(1-q)m_i^\alpha\Big)\delta(H_i^\alpha)$$
$$\times\delta\Big(Nq - \sum_i \mathbf{m}_i^2\Big)\,\delta\Big(Nf - \sum_i f(m\beta h_i)\Big).$$

Here $H_i^\alpha = m_i^\alpha - \widehat{h}_i^\alpha L(m\beta h_i)$ and the matrix $\mathbf{K}$ is such that $K_{ij}^{\alpha\beta} = \partial H_i^\alpha/\partial m_j^\beta = \sum_\gamma C_i^{\alpha\gamma}A_{ij}^{\gamma\beta}$, where

$$C_i^{\alpha\beta} = P_i^{\alpha\beta}\left(\frac{L(m\beta h_i)}{\beta h_i}\right) + \widehat{h}_i^\alpha\widehat{h}_i^\beta mL'(m\beta h_i) \tag{6}$$

with $P_i^{\alpha\beta} = \delta^{\alpha\beta} - \widehat{h}_i^\alpha\widehat{h}_i^\beta$, and

$$A_{ij}^{\alpha\beta} = \delta_{ij}(C_i^{-1})^{\alpha\beta} - \delta^{\alpha\beta}\left[\beta J_{ij} - \beta^2(1-q)\delta_{ij}\right] - \frac{2\beta^2}{N}m_i^\alpha m_j^\beta. \tag{7}$$

The matrix $\mathbf{A}$ is just the Hessian matrix of the TAP free energy given by $A_{ij}^{\alpha\beta} = \partial^2(\beta F/m)/\partial m_i^\alpha\partial m_j^\beta$.

Averaging the number of solutions (5) over the J_{ij} distribution, we obtain [14]

$$\frac{1}{N}\ln\langle N_s(f)\rangle_J = -m\Lambda q - uf - \frac{m}{2\beta^2}\Delta^2 - m(1-q)\Delta + \ln I, \tag{8}$$

where

$$I = \frac{2}{\Gamma(\frac{m}{2})}\left(\frac{m}{2q}\right)^{\frac{m}{2}}\int_0^\infty dh\, h^{m-1}\exp\Big[m\Lambda L^2(m\beta h) + uf(m\beta h) - \frac{m}{2\beta^2 q}\Big(\beta h - \Delta L(m\beta h)\Big)^2\Big]. \tag{9}$$

The parameters q, Λ, Δ and u were originally introduced as integration variables and are fixed according to the steepest descent method to make the right hand side of Eq. (8) stationary. The corresponding saddle point equations for these variables can be derived in a straightforward way. We have solved them numerically for temperatures $T < T_c$. The calculation for the total number of solutions $\langle N_S\rangle_J \sim e^{N\Sigma(T)}$ is done by setting $u = 0$. We have evaluated $\Sigma(T)$ as a function of temperature T for the XY and Heisenberg cases. We find that $\Sigma(T)$ increases with decreasing T, approaching the known $T = 0$ values, $\Sigma(0) = 0.02328$ for $m = 2$ and $\Sigma(0) = 0.00839$ for $m = 3$ [14] and vanishes as T approaches T_c.

We now turn to the determination of the eigenvalues of the Hessian matrix $\mathbf{A}$. In Eq. (6), both $P_i^{\alpha\beta}$ and $\widehat{h}_i^\alpha\widehat{h}_i^\beta$ are projection operators, therefore, the eigenvalue spectrum of $C_i^{\alpha\beta}$ is simple: one eigenvalue $mL'(m\beta h_i)$ and $(m-1)$ eigenvalues $L(m\beta h_i)/\beta h_i$. They are both positive. For convenience, we denote by $B_{ij}^{\alpha\beta}$ the part of the matrix $\mathbf{A}$ without the final $O(1/N)$ term in (7): $A_{ij}^{\alpha\beta} = B_{ij}^{\alpha\beta} - \frac{2\beta^2}{N}m_i^\alpha m_j^\beta$. The $O(1/N)$ term has the form of a projection operator and does not contribute to the extensive part of the complexity $\Sigma(T)$. For the computation of $\Sigma(T)$, we can drop the the modulus sign around $\mathbf{K}$ in (5) [1]. We will find below that, as in the Ising spin glass case, the projector term produces an isolated eigenvalue of value zero [1].

We calculate the eigenvalue density $\rho(\lambda)$ of the Hessian matrix $\mathbf{B}$ without the projector term using the resolvent matrix

$$\mathbf{G}(\lambda) = (\lambda\mathbf{I} - \mathbf{B})^{-1}, \tag{10}$$

where $\mathbf{I}$ is the $(mN \times mN)$ unit matrix. We can write

$$\rho(\lambda) = \frac{1}{N\pi}\,\mathrm{Im}\;\mathrm{Tr}\;\mathbf{G}(\lambda - i\delta), \tag{11}$$

where δ is a positive infinitesimal. We use the locator expansion method [16] to calculate $\rho(\lambda)$ in the thermodynamic limit. The eigenvalue spectrum consists of two branches, transverse and longitudinal, which are denoted respectively by ρ_t and ρ_l. We can write

$$\rho(\lambda) = (m-1)\rho_t(\lambda) + \rho_l(\lambda). \tag{12}$$

Each branch is obtained from the corresponding resolvent matrices: $\rho_t(\lambda) = \pi^{-1}\mathrm{Im}\;\bar{G}_t(\lambda - i\delta)$ and $\rho_l(\lambda) = \pi^{-1}\mathrm{Im}\;\bar{G}_l(\lambda - i\delta)$, where

$$\frac{1}{N}\mathrm{Tr}\;\mathbf{G}(\lambda) \equiv \bar{G}(\lambda) = (m-1)\bar{G}_t(\lambda) + \bar{G}_l(\lambda). \tag{13}$$

The functions $\bar{G}_t(\lambda)$ and $\bar{G}_l(\lambda)$ satisfy

$$\bar{G}_t(\lambda) = \left\langle\left[\lambda - \beta^2(1-q) - \frac{\beta h}{L(m\beta h)} - \beta^2\bar{G}(\lambda)\right]^{-1}\right\rangle, \tag{14}$$

$$\bar{G}_l(\lambda) = \left\langle\left[\lambda - \beta^2(1-q) - \frac{1}{mL'(m\beta h)} - \beta^2\bar{G}(\lambda)\right]^{-1}\right\rangle, \tag{15}$$

where the averages were evaluated over the weight function in the integrand of Eq. (9). In order to solve the

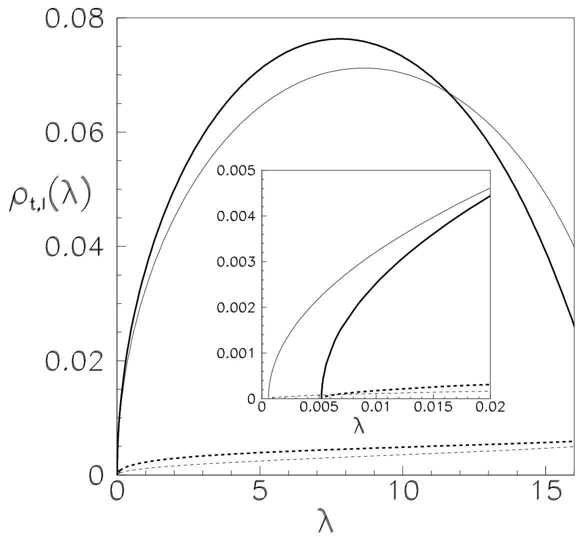

FIG. 1: The transverse and longitudinal eigenvalue densities, $\rho_t(\lambda)$ (solid lines) and $\rho_l(\lambda)$ (dashed lines), respectively, of the Hessian matrix for $T = 0.2$ and $u = 0$ (excluding the delta functions associated with null eigenvalues). Thick lines correspond to $m = 2$ (XY spins), while thin lines are for $m = 3$ (Heisenberg spins). The inset shows the behavior of these quantities at small values of λ, which clearly indicate the presence of finite gaps in the spectrum.

above equations, we separate $\bar{G}_t(\lambda)$ and $\bar{G}_l(\lambda)$ into real and imaginary parts. The resulting four coupled equations are solved numerically to yield $\rho_t(\lambda)$ and $\rho_l(\lambda)$ for given temperature T. The results for $T = 0.2$ are shown in Fig. 1. We find that, as T approaches T_c, $\rho_t(\lambda)$ and $\rho_l(\lambda)$ become equal, but in the low temperature limit $T \to 0$, however, the longitudinal eigenvalues become harder and harder. This is already apparent at $T = 0.2$ in Fig. 1, where ρ_t is much larger than ρ_l except at large values of λ. One of our main findings is that, for $0 < T < T_c$ and for finite m, the continuous part of the eigenvalue spectrum does not extend to $\lambda = 0$ as there is a very small but finite gap which can be seen in the inset of Fig. 1. This is in contrast to the behavior at zero-temperature of the same model [15], where the lower band edge is at $\lambda = 0$ for all m, and the large-m limit where explicit solution of our equations shows that the spectrum is gapless at all temperatures below T_c and given by the well-known semi-circle law: $\rho_{m\to\infty}(\lambda)/m = \sqrt{\lambda(4\beta - \lambda)}/(2\pi\beta^2)$.

For λ small, we can obtain analytic solutions to Eqs. (14) and (15). Expanding these equations in powers of $\lambda - \beta^2[\bar{G}(\lambda) - \bar{G}(0)]$, we find that both longitudinal and transverse branches show a square-root dependence but with different overall coefficients. We find

$$\rho_{t,l}(\lambda) \sim \frac{c_{t,l}}{\pi\sqrt{p}}\sqrt{\lambda - \frac{x_p^2}{4p}}, \tag{16}$$

where

$$\begin{aligned} x_p &= 1 - \beta^2\Bigg[\left(1 - \frac{1}{m}\right)\langle\left(\frac{L(m\beta h)}{\beta h}\right)^2\rangle \\ &\qquad + \frac{1}{m}\langle(mL'(m\beta h))^2\rangle\Bigg], \end{aligned} \tag{17}$$

$$\begin{aligned} p &= \beta^2\Bigg[\left(1 - \frac{1}{m}\right)\langle\left(\frac{L(m\beta h)}{\beta h}\right)^3\rangle \\ &\qquad + \frac{1}{m}\langle(mL'(m\beta h))^3\rangle\Bigg], \end{aligned} \tag{18}$$

and

$$c_t = \langle\left(\frac{L(m\beta h)}{\beta h}\right)^2\rangle + \frac{x_p}{p}\langle\left(\frac{L(m\beta h)}{\beta h}\right)^3\rangle, \tag{19}$$

$$c_l = \langle(mL'(m\beta h))^2\rangle + \frac{x_p}{p}\langle(mL'(m\beta h))^3\rangle. \tag{20}$$

Note that $(m-1)c_t + c_l = m/\beta^2$ so that the total density $\rho(\lambda) \sim m\sqrt{\lambda - x_p^2/(4p)}/(\pi\beta^2\sqrt{p})$. Our numerical solutions for small λ agree quite well with the square-root behavior. In the limit $m \to 1$, $m_i = L(\beta h_i) = \tanh(\beta h_i)$, and therefore $x_p = 1 - \beta^2\langle(1 - m_i^2)^2\rangle$ and $p = \beta^2\langle(1 - m_i^2)^3\rangle$ – the known results for Ising spin glasses [17].

We have directly evaluated the parameter which determines the gap in the expression (16). In order to compare with the $T = 0$ result, we consider the Hessian obtained from differentiating F/m instead of $\beta F/m$. The eigenvalues will then be scaled as $\lambda' = \lambda/\beta$, and therefore, the gap in the small-λ' spectrum is now determined by the gap parameter $\frac{x_p^2}{4\beta p}$. This quantity was evaluated as a function of temperature with the results shown in Fig. 2. It vanishes both at $T = 0$ and $T = T_c$ as expected. The figure also suggests that the gap becomes even smaller when the number of components m gets larger, which is consistent with the result of no gap at all in the large-m limit. Based on the behavior of the Ising case [3], we would expect x_p to tend to zero for the TAP states whose free energies approach those of the pure states. In other words, the TAP solutions with the lowest free energy will be gapless at all temperatures.

At first sight, the appearance of a finite gap in the eigenvalue spectrum at finite temperature is rather puzzling, since the continuous rotational symmetry of the vector spin glass model would suggest the existence of null eigenvalues associated with a uniform rotation of all the spins. One can indeed show that the Hessian matrix in the form of Eq. (7) admits $(m-1)$ null eigenvectors in the transverse sector as follows: note first that $(C_i^{-1})^{\alpha\beta} = (\delta^{\alpha\beta} - \widehat{m}_i^\alpha\widehat{m}_i^\beta)K(m_i)/m_i + \widehat{m}_i^\alpha\widehat{m}_i^\beta K'(m_i)$, where $K(x) = L^{-1}(x)/m$. For $(m-1)$ vectors $\boldsymbol{\xi}_i$ satisfying $\sum_i \boldsymbol{\xi}_i \cdot \widehat{\mathbf{m}}_i = 0$ and $\xi_i = m_i$, it is easy to show that $\sum_{j,\beta} A_{ij}^{\alpha\beta}\xi_j^\beta = 0$ using the defining TAP equations.

The analysis of the effect of the $O(1/N)$ projector term in the Hessian matrix closely follows that for the Ising

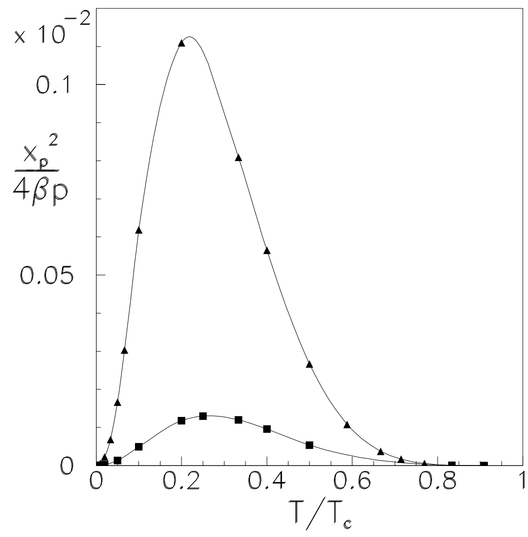

FIG. 2: The parameter $x_p^2/4\beta p$ that determines the size of the gap in the Hessian spectrum as a function of temperature T for $m = 2$ (triangles) and $m = 3$ (squares).

case [1]. There is an isolated null eigenvalue outside the main band in the longitudinal sector if

$$\frac{2\beta^2}{N}\sum_{i,j}\sum_{\alpha,\beta} m_i^\alpha (B^{-1})_{ij}^{\alpha\beta} m_j^\beta \equiv 2\beta^2 H = 1 \tag{21}$$

with the eigenvector $v_i^\alpha = \sum_{j,\beta}(B^{-1})_{ij}^{\alpha\beta} m_j^\beta$. We evaluate H using the expression

$$\begin{aligned}(B^{-1})_{ij}^{\alpha\beta} &= \sqrt{\det \mathbf{B}} \int \prod_{i,\alpha}\left(\frac{d\phi_i^\alpha}{\sqrt{2\pi}}\right)\phi_i^\alpha \phi_j^\beta \\ &\quad \times \exp\Big[-\frac{1}{2}\sum_{i,j}\sum_{\alpha,\beta}\phi_i^\alpha B_{ij}^{\alpha\beta}\phi_j^\beta\Big] \end{aligned}\tag{22}$$

then insert H into the integrand of (5). The rest of the calculation involves the evaluations of various Gaussian integrals and can be done as for the Ising case [1], except that there are extra indices for the vector components. We obtain

$$H = \frac{q^2 A_3}{(A_1 - q)^2 - A_3(A_2 - q(1-q))}, \tag{23}$$

where

$$A_1 = \langle L(m\beta h)\{\beta h - \Delta L(m\beta h)\} m L'(m\beta h)\rangle, \tag{24}$$

$$A_2 = \langle \{\beta h - \Delta L(m\beta h)\}^2 m L'(m\beta h)\rangle, \tag{25}$$

$$A_3 = \langle L^2(m\beta h) m L'(m\beta h)\rangle. \tag{26}$$

The averages are evaluated with respect to the integrand of (9). The expression for H is exactly the same as the one for the Ising spin glass case (Eq. (10) of Ref. [1]) but with the quantities A_1, A_2, and A_3 generalized to the m-vector case. The numerical evaluation of the required integrals reveals that $2\beta^2 H = 1$ for all $T \le T_c$ and at all values of u.

Therefore, in the thermodynamic limit, the Hessian matrix of the m-vector TAP free energy is positive semidefinite with exactly m eigenvalues equal to zero; the vanishing of $(m-1)$ of these eigenvalues is a consequence of rotational invariance but the vanishing of the isolated longitudinal eigenvalue indicates that the TAP states for vector spin glasses will behave like those for Ising spin glasses. This means that there will be only minima and index-one saddles at large but finite values of N. Dynamically the vector spin glasses will behave more like the Ising spin glass than p-spin models. For example a state with remanent magnetization will evolve in time to a state with zero magnetization for all the vector glasses, whereas for p-spin models, remanent magnetization remains forever – all because the existence of the null longitudinal eigenvalue means that the TAP states in vector spin glasses are just turning points of the free energy and so are unable to trap the system in their vicinity.

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